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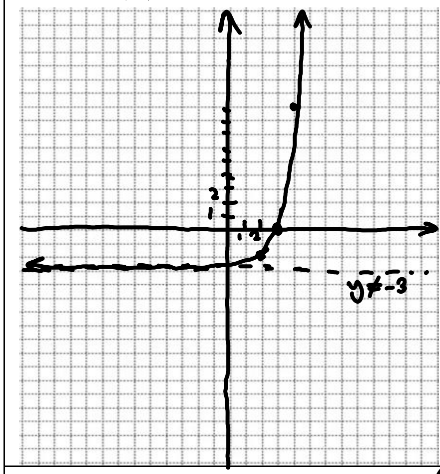
Date: Feb 23rd, 2024

**Math 12 Honours: Section 5.4 Graphing Exponential & Logarithmic Equations with Transformations**

1. For each graph below, find the Y-intercept, X-intercept (if any), Domain and Range, and Asymptotes. Then graph the function with the grid provided. Be sure to label the axis on your grid.

a) $y=3(2)^{x-2}-3$




y-int: $3(2)^{-2}-3$
$=-\frac{9}{4}$

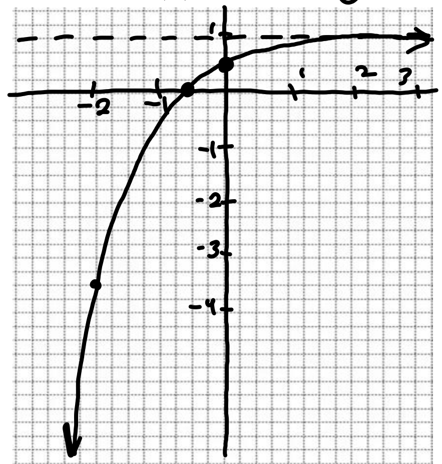
x-int: $3=3(2)^{x-2}$
$x=2$

D: $x\in\mathbb{R}$

R: $y>-3$

Asymptote: $y\neq-3$

b) $y=-0.5(3)^{-x}+1 \Rightarrow y=-0.5\left(\frac{1}{3}\right)^{x}+1$



y-int: $y=-0.5(1)+1=\frac{1}{2}$

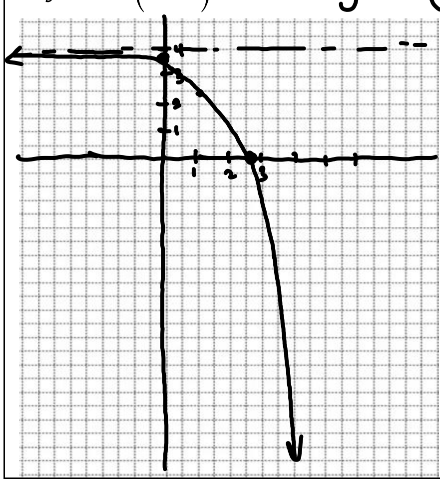
x-int: $\log_{\frac{1}{3}}2=-0.63$

D: $x\in\mathbb{R}$

R: $y>1$

$y\neq1$

c) $y=-2\left(0.\overline{33}\right)^{2-x}+4$ $\quad y=-2(3)^{x-2}+4$



y-int: $\frac{34}{9}$

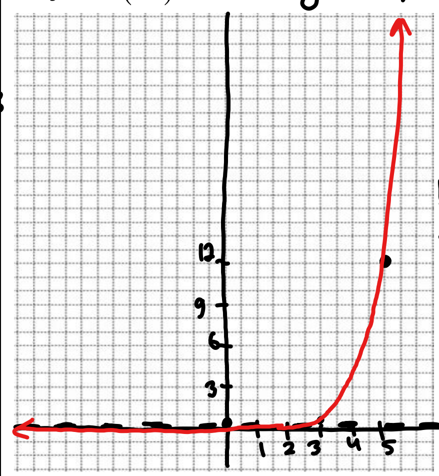
x-int: $\log_3 2+2=2.63$

D: $x\in\mathbb{R}$

R: $y<4$

Asymptote: $y\neq4$

d) $y=4(81)^{0.25x-1}$ $\quad y=4(3)^{x-4}$



y-int: $4(3)^{-4}=0.049$

x-int: $\log_3 0+4=\varnothing$

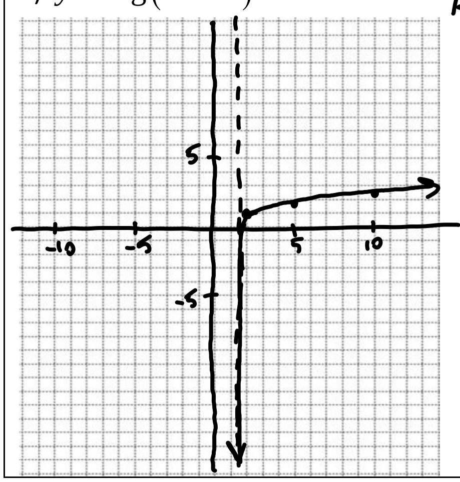
D: $x\in\mathbb{R}$

R: $y>0$

asymptote: $y\neq0$

2. Graph each of the following logarithmic functions. Indicate the Domain, Range, equations of Asymptotes, and any intercepts. Be sure to label your axis on the graph:

a) $y=\log(2x-3)+1$



Asymptote:
$2x-3\neq0$
$x\neq\frac{3}{2}$

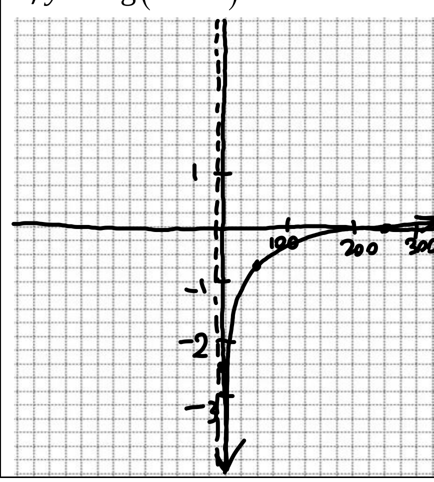
D: $x>\frac{3}{2}$

R: $y\in\mathbb{R}$

y-int: $\varnothing$

x-int: 1.55

b) $y=\log(4x+1)-3$



Asymptote:
$4x+1\neq0$
$x\neq-\frac{1}{4}$

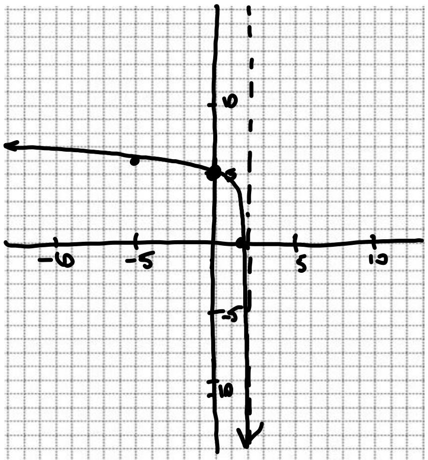
y-int: $\log(5)-3$
$=-2.3$

x-int:
$4x+1=10^3$
$x=249.75$

D: $x>-\frac{1}{2}$

R: $y\in\mathbb{R}$

c) $y=\log_2(2-x)+4$

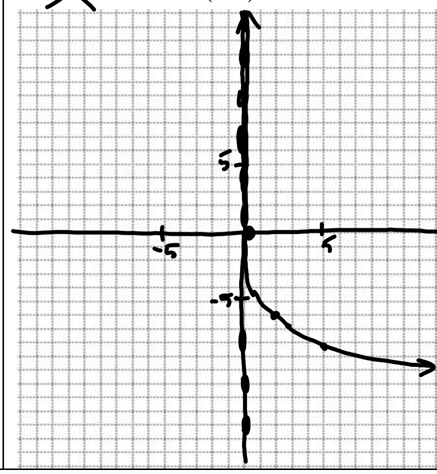


Asymptote:
$2-x\neq 0$
$x\neq 2$

y-int: $\log_2 2+4$
$=5$

x-int: $2^{-4}=2-x$
$=1.9375$

d) $\cancel{\log}\; y=-\log_3(3x)-5$



Asymptote:
$x\neq 0$

y-int: $\varnothing$

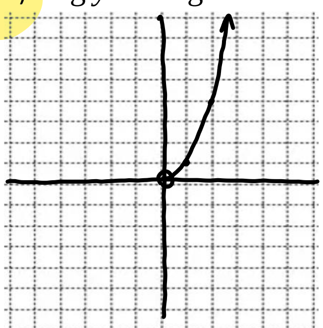
x-int: $3^{-5}=3x$
$x=0.00137$

D: $x>0$

R: $y\in\mathbb{R}$

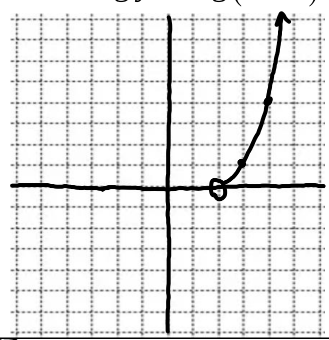
3. Graph the following functions. Indicate the domain and range:

a) $\log y=2\log x$



$\boxed{\begin{array}{l}D:\ x>0\\ R:\ y>0\end{array}}$

b) $0.5\log y=\log(x-2)$



$\sqrt{y}=x-2$

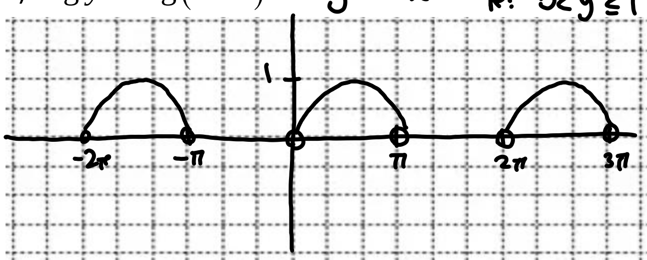
$y=(x-2)^2$

$\boxed{\begin{array}{l}D:\ x>2\\ R:\ y>0\end{array}}$

c) $\log y=\log(\sin x)$ $y=\sin x$

D: $2n<x<2n+1;n\in\mathbb{Z}$

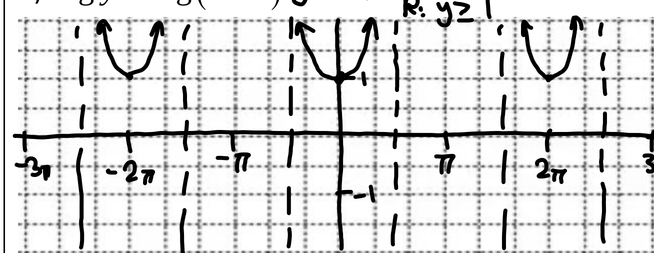
R: $0<y\leq 1$



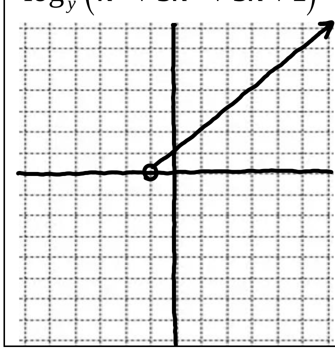
d) $\log y=\log(\sec x)$ $y=\sec x$

D: $\left(\frac{2n+1}{2}\right)\pi<x<\left(\frac{2n+3}{2}\right)\pi\ ;\ n\in\mathbb{Z}$

R: $y\geq 1$



$\log_y\left(x^3+3x^2+3x+1\right)=3$



$y^3=x^3+3x^2+3x+1$

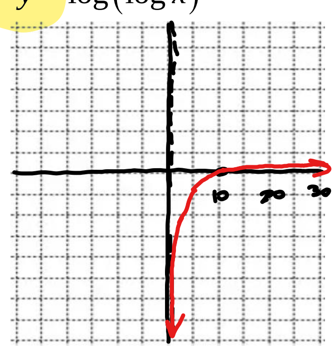
$y^3=(x+1)^3$

$y=x+1$

D: $x>-1$

R: $y>0$

$y=\log(\log x)$



$x\neq 0$

y-int: $\varnothing$

x-int: 10

D: $x>1$

R: $y\in\mathbb{R}$

4. Given that $f(x)=\log_3(x+2)-4$, find $f^{-1}(x)$, the inverse function of $f(x)$

$y=\log_3(x+2)-4 \Rightarrow x=\log_3(y+2)-4 \Rightarrow 3^{x+4}=y+2 \Rightarrow y=3^{x+4}-2$

$$\boxed{f^{-1}(x)=3^{x+4}-2}$$

5. What transformation is required to go from $y=\log x$ to $y=\log\left(\frac{1}{x}\right)$ ?

$x \to \frac{1}{x}$ or $y=\log x \to y=-\log x$

Reciprocal $\boxed{\text{Vertical Reflection}}$

6. What transformation is required to go from $y=\log_3 x$ to $y=\log_3 \frac{4}{x}$ ?

$y=\log_3 x \to y=-\log_3 x \to y=-\log_3 x + \log_3 4$

Vertical Reflection and Vertical shift by $\log_3 4$ up.

7. Are the following graphs the same? Yes or NO? Explain:

a) $y=4(0.5)^x$ and $y=4(2)^{-x}$

yes

$(2^{-1})^x = (0.5)^x$

b) $y=24(0.5)^{2-x}$ and $y=6(2)^x$

$y=24(2)^{x-2} = 6(2)^{x-2+2}$

yes

8. What is the inverse function of $f(x)=3(5)^{x-2}$ ? What are the domain, range, x-intercepts, and Y-intercepts of both $f(x)$ and $f^{-1}(x)$ ? What patterns do you notice?

$y=3(5)^{x-2}$

$x=3(5)^{y-2}$

$y-2=\log_5 \frac{x}{3}$

$\boxed{f^{-1}(x)=\log_5 \frac{x}{3}+2}$

$f(x)$: D: $x\in\mathbb{R}$ x-int: $\varnothing$
R: $y>0$ y-int: $\frac{3}{25}$

$f^{-1}(x)$ = D: $x>0$ x-int: $\frac{3}{25}$
R: $y\in\mathbb{R}$ y-int: $\varnothing$

all switched as expected b/c inverse function!

9. Find the coordinates of the points of intersection of the graphs: $y=\log_{10}(x-2)$ and $y=1-\log_{10}(x+1)$ (Euclid)

$\log(x-2)=1-\log(x+1)$

$\log(x-2)=\log\left(\frac{10}{x+1}\right)$

$x-2=\frac{10}{x+1} \Rightarrow x^2-x-2=10$

$x^2-x-12=0$

$(x-4)(x+3)=0 \Rightarrow \boxed{x_1=4}$ $\cancel{x_2=-3}$ rej.

$y=\log_{10}(2)$

$\boxed{(x,y)=(4,\log_{10}2)}$

10. Determine all the points where the two functions intersect: $y=\log_{10}x^4$ and $y=(\log_{10}x)^3$ (Euclid)

$\log x^4=(\log x)^3$

$4\log x=(\log x)^3$

$\log x\left[4-(\log x)^2\right]=0$

$\log x=0 \Rightarrow x_1=1$ $\quad \log x=\pm 2 \Rightarrow x_2=100,\ x_3=\frac{1}{100}$

$y_1=0$

$y_2=8$

$y_3=-8$

$\boxed{(x,y)=(1,0),(100,8),\left(\frac{1}{100},-8\right)}$

11. If $\log(a+b)=x$ and $\log(a^2-ab+b^2)=y$ , then what is the value of $\sqrt[4]{a^3+b^3}$ in terms of "x" and "y"?

$\sqrt{a^3+b^3}=\sqrt[4]{(a+b)(a^2-ab+b^2)}=\sqrt[4]{10^x\cdot 10^y}=\sqrt[4]{10^{x+y}}=\boxed{10^{\frac{x+y}{4}}}$

$a+b=10^x$

$a^2-ab+b^2=10^y$

12. Solve the inequality

a) $\log_4(2x-3) > 2$

$2x-3 > 4^2$

$\boxed{x > \frac{19}{2}}$

b) $\log_{\frac{1}{4}} x > 5$

$x < \left(\frac{1}{4}\right)^5$

Sign depends on log base!

$\boxed{x < \frac{1}{1024}}$

13. What is the domain of the following functions:

i) $y = \log_{0.5}(\log_5 x)$

$\log_5 x > 0$

$\boxed{x > 1}$

ii) $y = \log_{0.5}\left(\log_5\left(\log_{\frac{1}{3}} x\right)\right)$

$\log_5\left(\log_{\frac{1}{3}} x\right) > 0$

$\log_{\frac{1}{3}} x > 1$

$\boxed{x < \frac{1}{3}}$

14. Solve for "x": $\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$ (Euclid)

$\log_4 x = \frac{7}{6} + \log_x 2$

let $\log x = A$

$\frac{A}{\log 4} = \frac{7}{6} + \frac{\log 2}{A} \Rightarrow 6A^2 = 14A\log 2 + 12(\log 2)^2$

$6A^2 - 14A\log 2 - 12(\log 2)^2$

$3A^2 - 7A\log 2 - 6(\log 2)^2$

$(A - 3\log 2)(3A + 2\log 2) = 0$

$\log x = \log 2^3$ $\quad$ $\log x^3 = \log 4$

$\boxed{x = 2^3}$ $\quad$ $\boxed{x = \sqrt[3]{4}}$

15. Find all the values of "x" such that: $\log_{2x}\left(48\sqrt[3]{3}\right) = \log_{3x}\left(162\sqrt[3]{2}\right)$ (Euclid)

$A = \log x$

$B = \log 2$

$C = \log 3$

$\frac{\log\left(3^{\frac{4}{3}} \cdot 2^4\right)}{\log 2x} = \frac{\log\left(2^{\frac{4}{3}} \cdot 3^4\right)}{\log 3x}$

$(\log 2 + \log x)\left(\frac{4}{3}\log 2 + 4\log 3\right) = (\log 3 + \log x)\left(\frac{4}{3}\log 3 + 4\log 2\right)$

$(B+A)\left(\frac{4}{3}B + 4C\right) = (C+A)\left(\frac{4}{3}C + 4B\right)$

$\frac{4}{3}B^2 + \cancel{4BC} + \frac{4}{3}AB + 4AC = \frac{4}{3}C^2 + \cancel{4BC} + \frac{4}{3}AC + 4AB \Rightarrow \frac{4}{3}B^2 - \frac{4}{3}C^2 = \frac{8}{3}AB - \frac{8}{3}AC$

$4B^2 - 4C^2 = 8A(B-C)$

$2A = \frac{\cancel{(B-C)}(B+C)}{\cancel{(B-C)}}$

$A = \frac{B+C}{2}$

$A^2 = \log 6$

$\boxed{x = \sqrt{6}}$

16. Determine all real numbers "x" for which: $2\log_2(x-1) = 1 - \log_2(x+2)$ (Euclid)

$\log_2(x-1)^2 = 1 - \log_2(x+2)$

$\log_2(x-1)^2 = \log_2\frac{2}{x+2}$

$(x-1)^2 = \frac{2}{x+2}$

$(x^2 - 2x + 1)(x+2) = 2$

$x^3 + \cancel{2x^2} - \cancel{2x^2} - 4x + x + 2 = 2$

$x^3 - 3x = 0$

$x(x^2 - 3) = 0$

$x = \cancel{0}, \sqrt{3}, \cancel{-\sqrt{3}}$ rej rej

$\boxed{x = \sqrt{3}}$

17. Determine all pairs of angles (x,y) with $0° \le x < 180°$ and $0° \le y < 180°$ that satisfy the following systems of equations: (Euclid)

$$\log_2(\sin x \cos y) = -\frac{3}{2} \quad \text{and} \quad \log_2\left(\frac{\sin x}{\cos y}\right) = \frac{1}{2}$$

$\log_2(\sin x \cos y) + \log_2\left(\frac{\sin x}{\cos y}\right) = -1 \Rightarrow \log_2 \sin^2 x = -1 \Rightarrow \sin^2 x = \frac{1}{2} \Rightarrow \sin x = \sqrt{\frac{1}{2}}$

$x_1 = 45°$

$\log_2\left(\frac{\cos y}{\sqrt{2}}\right) = -\frac{3}{2}$

$\cos y = 2^{-\frac{3}{2}} \cdot \sqrt{2}$

$\cos y = \frac{1}{2} \Rightarrow y_1 = 60°$

$\boxed{(x,y) = (45°, 60°), (135°, 60°)}$

18. Determine all pairs (a,b) of real numbers that satisfy the following systems of equations: Give your answer in simplified exactd form: (Euclid)

$$\sqrt{a} + \sqrt{b} = 8 \quad \text{and} \quad \log_{10} a + \log_{10} b = 2$$

$a, b \ge 0$

① $a + 2\sqrt{ab} + b = 64$

② $\log_{10} ab = 2 \Rightarrow \underline{ab = 100}$

③ $a + 20 + b = 64$

$a + b = 44$

$\underline{a = 44 - b}$

④ $(44-b)b = 100 \Rightarrow b^2 - 44b + 100 = 0$

$b = \frac{44 \pm \sqrt{44^2 - 400}}{2}$

$b = 22 \pm 8\sqrt{6}$

$a = 22 \pm 8\sqrt{6}$

$\boxed{(a,b) = (22+8\sqrt{6}, 22-8\sqrt{6}), (22-8\sqrt{6}, 22+8\sqrt{6})}$

19. Solve for "x" $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ (AIME)

$\frac{20}{x} \cancel{\log_2 3} = \frac{2020}{x+3} \cancel{\log_2 3}$

$\frac{1}{x} = \frac{101}{x+3} \Rightarrow x + 3 = 101x$

$100x = 3$

$\boxed{x = \frac{3}{100}}$

## Problem

The value of $x$ that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

## Solution

Let $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020} = n$. Based on the equation, we get $(2^x)^n = 3^{20}$ and $(2^{x+3})^n = 3^{2020}$. Expanding the second equation, we get $8^n \cdot 2^{xn} = 3^{2020}$. Substituting the first equation in, we get $8^n \cdot 3^{20} = 3^{2020}$, so $8^n = 3^{2000}$. Taking the 100th root, we get $8^{\frac{n}{100}} = 3^{20}$. Therefore, $(2^{\frac{3}{100}})^n = 3^{20}$, and using the our first equation($2^{xn} = 3^{20}$), we get $x = \frac{3}{100}$ and the answer is $\boxed{103}$. ~rayfish