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Math 12 Honours: Section 5.3 Solving Logarithmic Equations with Identities

1. Solve for "x" and state all the extraneous roots: Show your work:

a) $\log(6-x)-2\log x=0$ $\log\frac{6-x}{x^2}=0$ $\frac{6-x}{x^2}=1$ $x^2+x-6=0$ $(x-2)(x+3)=0$ $\boxed{x_1=2}$ $\cancel{x_2=-3}$ rej. $x<0$ Restrictions: $6-x>0$, $x<6$ $x<0$	b) $\log_3 x^3-\log_3 3x=3$ $\log_3\frac{x^3}{3x}=3$ $\frac{x^2}{3}=27$ $x^2=81$ $\boxed{x_1=9}$ $\cancel{x_2=-9}$ rej. $x>0$ Restrictions: $x^3>0$ $3x>0$ $\Rightarrow$ $x>0$
c) $\log_2(x-3)+\log_2(x+1)=5$ $\log_2(x-3)(x+1)=5$ $32=x^2-2x-3$ $x^2-2x-35=0$ $(x-7)(x+5)=0$ $\boxed{x_1=7}$ $\cancel{x_2=-5}$ rej. $x>3$ Restrictions: $x-3>0$ $x>3$ ← more restrictive $x+1>0$ $x>-1$	d) $\log_7(x+1)+\log_7 x=\log_7 12$ $\log_7 x(x+1)=\log_7 12$ $x^2+x=12$ $x^2+x-12=0$ $(x-3)(x+4)=0$ $\boxed{x_1=3}$ $\cancel{x_2=-4}$ rej. $x>0$ Restrictions: $x+1>0$ $x>-1$ $x>0$
e) $\log_4(16x-64)-\log_4(3x-34)=3$ $\log_4\frac{16x-64}{3x-34}=3$ $64=\frac{16x-64}{3x-34}$ $12x-136=x-4$ $11x=132$ $\boxed{x=12}$ Restrictions: $16x-64>0$ $x>4$ $3x-34>0$ $x>11\frac{1}{3}$	f) $\log(x-7)+\log(x+2)=1$ $\log(x-7)(x+2)=1$ $x^2-5x-14=10$ $x^2-5x-24=0$ $(x-8)(x+3)=0$ $\boxed{x_1=8}$ $\cancel{x_2=-3}$ rej $x>7$ Restrictions $x-7>0$ $x+2>0$ $x>7$ $x>-2$
g) $\log_3(3x+2)+\log_3 x=\log_3 56$ $\log_3 x(3x+2)=\log_3 56$ $3x^2+2x=56$ $3x^2+2x-56=0$ $(x-4)(3x+14)=0$ $\boxed{x_1=4}$ $\cancel{x_2=\frac{-14}{3}}$ rej. $x>0$ Restrictions $3x+2>0$ $x>-\frac{2}{3}$ $x>0$	h) $\log x=\frac{2}{3}\log 27+2\log 2-\log 3$ $\log x=\log\frac{9\cdot 4}{3}$ $\boxed{x=12}$ Restrictions: $x>0$

i) $2\log_4 x+\log_4(x-2)-\log_4 2x=1$

$\log_4 \frac{x^2(x-2)}{2x}=1$

$\frac{x^2-2x}{2}=4$

$x^2-2x-8=0$

$(x-4)(x+2)=0$

$\boxed{x_1=4}$ $\quad \cancel{x_2=-2}$ rej $x>2$

Restrictions

$x>0$

$x-2>0$

$x>2$

j) $\log_2 16^{2x+1}=8$

$2^{8x+4}=2^8$

$8x+4=8$

$\boxed{x=\frac{1}{2}}$

Restrictions

$16^{2x+1}>0$

l) $(\log_8 a)(\log_a 3a)(\log_{3a} x^2)=\log_a a^5$

$\frac{\cancel{\log a}}{\log 8}\cdot\frac{\cancel{\log 3a}}{\cancel{\log a}}\cdot\frac{2\log x}{\cancel{\log 3a}}=5$

$\log x=\frac{5}{2}\log 8$

$x=8^{\frac{5}{2}}=\boxed{128\sqrt{2}}$

$x>0$

m) $2\log(3-x)=\log 2+\log(22-2x)$

$(3-x)^2=4(11-x)$

$9-6x+x^2=44-4x$

$x^2-2x-35=0$

$(x-7)(x+5)=0$

$\cancel{x_1=7}$ rej. $x<3$ $\quad \boxed{x_2=-5}$

Restrictions

$3-x>0$

$x\leq 3$

$22-2x>0$

$x<11$

n) $2^{\log x^2}=3\left(2^{1+\log x}\right)+16$

$2^{2\log x}=6\cdot 2^{\log x}+16$

$2^{\log x}=A$

$A^2-6A-16=0$

$(A-8)(A+2)=0$

$2^{\log x}=8$ $\quad \cancel{2^{\log x}=-2}$ rej.

$\log x=3$

$\boxed{x=1000}$

o) $\log_5(x+3)+\log_5(x-1)=1$

$\log_5 (x+3)(x-1)=1$

$5=x^2+2x-3$

$x^2+2x-8=0$

$(x-2)(x+4)=0$

$\boxed{x_1=2}$ $\quad \cancel{x_2=-4}$ rej. $x>1$

Restrictions

$x-1>0$

$x>1$

$x+3>0$

$x>-3$

$\log_2(x+2)+5=8+\log_2 x$

$\log_2 \frac{x+2}{x}=3$

$8=\frac{x+2}{x}$

$8x=x+2$

$7x=2$

$\boxed{x=\frac{2}{7}}$

Restrictions:

$x+2>0$

$x>-2$

$x>0$

$2\log x-\log(24-x)=\log 2$

$\log\frac{x^2}{24-x}=\log 2$

$\frac{x^2}{24-x}=2$

$48-2x=x^2$

$x^2+2x-48=0$

$(x-6)(x+8)=0$

$\boxed{x_1=6}$ $\quad \cancel{x_2=-8}$ rej.

Restrictions

$0<x<24$

$\log_2(2x+4)-\log_2(x-1)=3$ $\log_2 \frac{2x+4}{x-1}=3$ Restrictions: $x>1$ $8=\frac{2x+4}{x-1}$ $4x-4=x+2$ $3x=6$ $\boxed{x=2}$	$\log_4 x+\log_2\sqrt{x-2}=1+\log_{16}(x-1)^2$ $\log_2 \sqrt{x}\cdot\sqrt{x-2}=1+\log_2\sqrt{x-1}$ $\log_2 \frac{\sqrt{x}\cdot\sqrt{x-2}}{\sqrt{x-1}}=1$ Restrictions: $x>0$, $x>2$ $2=\frac{\sqrt{x(x-2)}}{\sqrt{x-1}}$ $4(x-1)=x^2-2x$ $4x-4=x^2-2x$ $x^2-6x+4=0$ $x=\frac{6\pm\sqrt{36-16}}{2}=3\pm\sqrt{5}$ $\boxed{x_1=3+\sqrt{5}}$ $\cancel{x_2=3-\sqrt{5}}$ rej.

2. Solve for "x": $\log_a b+\log_{a^2} b=\log_{a^3} b^x$

$$\log_a b+\frac{1}{2}\log_a b=\frac{x}{3}\log_a b$$

$$\frac{3}{2}=\frac{x}{3}$$

$$\boxed{x=\frac{9}{2}}$$

3. Solve for "x": $\log_4 x-\log_x 16=\frac{7}{6}-\log_x 8$ [Euclid]

$$\log_4 x-\frac{7}{6}=\log_x 2$$

$$\frac{\log x}{\log 4}-\frac{7}{6}=\frac{\log 2}{\log x}\Rightarrow 6(\log x)^2-7\log 4\cdot\log x=6\log 2\cdot\log 4$$

$$\underset{2\ 3}{6}(\log x)^2-14\log 2\log x-\underset{2\ -6}{12}(\log 2)^2=0$$

$$(2\log x-6\log 2)(3\log x+2\log 2)$$

$$2\log x=6\log 2\Rightarrow \boxed{x=8} \qquad 3\log x=-2\log 2\Rightarrow \boxed{x=\sqrt[3]{\tfrac{1}{4}}}$$

4. Determine all value(s) of "x" that satisfy the equation (find all the extraneous roots) [Euclid]

$$\log_{5x+9}(x^2+6x+9)+\log_{x+3}(\underset{5}{5}x^2+24x+\underset{3}{27})=4$$

$$\log_{(5x+9)}(x+3)^2+\log_{(x+3)}(x+3)(5x+9)=4$$

$$2\log_{5x+9}(x+3)+1+\log_{x+3}(5x+9)=4$$

$$\frac{2\log(x+3)}{\log(5x+9)}+\frac{\log(5x+9)}{\log(x+3)}=3$$

Let $\log(x+3)=A$

Let $\log(5x+9)=B$

$$\frac{2A}{B}+\frac{B}{A}=3\Rightarrow 2A^2+B^2=3AB$$

$$2A^2-3AB+B^2=0$$

$$(A-B)(2A-B)=0$$

$A=B$

$\log(x+3)=\log(5x+9)$

$x+3=5x+9$

$4x=-6$

$\boxed{x=-\frac{3}{2}}$

$2A=B$

$(x+3)^2=5x+9$

$x^2+6x+\cancel{9}=5x+\cancel{9}$

$x^2+x=0$

$\boxed{x_1=0,\ x_2=-1}$

5. If $\log_2 x$, $(1+\log_4 x)$, $\log_8 4x$ are consecutive terms of a geometric sequence, determine the possible value(s) of "x". Euclid 2009

Let $\log_2 x = A$

$$\frac{1+\log_4 x}{\log_2 x} = \frac{\log_8 4x}{1+\log_4 x} \Rightarrow \frac{1+\frac{1}{2}\log_2 x}{\log_2 x} = \frac{\frac{2}{3}+\frac{1}{3}\log_2 x}{1+\frac{1}{2}\log_2 x}$$

$$\frac{1+\frac{1}{2}A}{A} = \frac{\frac{2}{3}+\frac{1}{3}A}{1+\frac{1}{2}A} \Rightarrow 1+A+\frac{1}{4}A^2 = \frac{2}{3}A+\frac{1}{3}A^2$$

$$\frac{1}{12}A^2 - \frac{1}{3}A - 1 = 0$$

$$A^2 - 4A - 12 = 0$$

$$(A-6)(A+2)=0$$

$$A_1 = 6 \Rightarrow \boxed{x=64} \quad A_2 = -2 \Rightarrow \boxed{x=\frac{1}{4}}$$

6. Find the product of the value(s) of "m" that satisfy this equation: $(\log_2 m)^2 + \log_2 m^3 = 10$ Brilliant

Let $\log_2 m = A$

$$A^2 + 3A - 10 = 0$$

$$(A-2)(A+5)=0$$

$$A_1 = 2 \quad A_2 = -5$$

$\log_2 m_1 = 2$ $\quad$ $\log_2 m_2 = -5$

$m_1 = 4$ $\quad$ $m_2 = -32$

$$\boxed{m_1 m_2 = -128}$$

7. Determine all real numbers "x" for which $\left(\sqrt{x}\right)^{\log_{10} x} = 100$ [Euclid]

$$\left(x^{\frac{1}{2}}\right)^{\log_{10} x} = 10^2$$

$$x^{\frac{1}{2}\log x} = 10^2$$

$$x^{\log_{100} x} = 100 \Rightarrow \log_{100} x = \log_x 100 \Rightarrow \frac{\log x}{2} = \frac{2}{\log x}$$

$$(\log x)^2 = 4$$

$$\log x = \pm 2$$

$$\boxed{x_1 = 100, x_2 = \frac{1}{100}}$$

8. Determine all real numbers "x" for which $(\log_{10} x)^{\log_{10}(\log_{10} x)} = 10{,}000$ [Euclid]

Let $A = \log_{10} x$

$$\log\left[(\log x)^{\log(\log x)}\right] = \log 10000$$

$$\log\left(A^{\log A}\right) = 4$$

$$\log A \cdot \log A = 4$$

$$\log A = \pm 2$$

$\log(\log x) = 2$ or $\log(\log x) = -2$

$\log x = 100$ $\quad$ $\log x = \frac{1}{100}$

$\boxed{x = 10^{100}}$ $\quad$ $\boxed{x = 10^{\frac{1}{100}}}$

9. Challenge: Given the two equations, find the value of "n" 2003 AIME

$$\log \sin x + \log \cos x = -1 \qquad \text{and} \qquad \log(\sin x + \cos x) = 0.5(\log n - 1)$$

$$\log(\sin x \cos x) = -1$$

$$\sin x \cos x = 10^{-1}$$

plug into

$$2\log(\sin x + \cos x) = \log n - 1$$

$$\log\left[(\sin x + \cos x)^2\right] + 1 = \log n$$

$$\log(\underbrace{\sin^2 x + \cos^2 x}_{=1} + \underbrace{2\sin x\cos x}_{=2\cdot 10^{-1}}) + 1 = \log n$$

$$\log\left(1+\frac{1}{5}\right) + 1 = \log n \Rightarrow \log n = \log\left(\frac{6}{5}\cdot 10\right)$$

$$\boxed{n = 12}$$

10. Determine the number of ordered pairs (a,b) of integers such that $\log_a b + 6\log_b a = 5$, with $2 \le a \le 2005$, and $2 \le b \le 2005$. 2005 AIME

$$\frac{\log b}{\log a} + \frac{6\log a}{\log b} = 5$$

$$\frac{B}{A} + \frac{6A}{B} = 5$$

$$B^2 + 6A^2 = 5AB$$

$$B^2 - 5AB + 6A^2 = 0$$

$$-2 \quad -3$$

$$(B-2A)(B-3A) = 0$$

$B = 2A$ | $B = 3A$

$\log b = \log a^2$ | $\log b = \log a^3$

$b = a^2$ **OR** $b = a^3$

$2 \le a \le 44$ | $2 \le a \le 12$

✓ $44^2 = 1936$

✗ $45^2 = 2025$

$43 + 11 = \boxed{54}$