COMPOSITE FUNCTIONS AND THEIR DOMAINS

A composite function is one in which the output of one function becomes the input for another. The notation is $(f \circ g)(x)$ or $f(g(x))$, read "f of g of x", where f and g are both functions of x. In $f(g(x))$, **the g(x) function is substituted for x in the f(x) function**. Similarly, in $g(f(x))$, **the f(x) function is substituted for x in the g(x) function**.

Examples:

1. If $f(x) = 2x + 1$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x)$ or $f(g(x)) = 2\sqrt{x} + 1$.

For $g(x) = \sqrt{x}$ and $f(x) = 2x + 1$, $(g \circ f)(x)$ or $g(f(x)) = \sqrt{2x+1}$

2. If $f(x) = x^2 + 2$ and $g(x) = \sqrt{3-x}$, then $(f \circ g)(x)$ or $f(g(x)) = \left(\sqrt{3-x}\right)^2 + 2 = 5 - x$.

For $g(x) = \sqrt{3-x}$ and $f(x) = x^2 + 2$, $(g \circ f)(x)$ or $g(f(x)) = \sqrt{3-(x^2+2)} = \sqrt{1-x^2}$

Finding the <u>domain</u> of a composite function consists of two steps:

Step 1. Find the domain of the "inside" (input) function. If there are any restrictions on the domain, <u>keep them</u>.

Step 2. Construct the composite function. Find the domain of this new function. If there are restrictions on this domain, <u>add them to the restrictions from Step 1</u>. If there is an overlap, use the more restrictive domain (or the intersection of the domains). The composite may also result in a domain unrelated to the domains of the original functions.

Examples:

1. In Example 2 above, the domain for $g(x) = \sqrt{3-x}$ is $x \leq 3$.
The domain for $f(g(x)) = 5 - x$ is all real numbers, but you must keep the domain of the inside function. So the domain for the composite function is also $x \leq 3$.

2. Also in Example 2, the domain for $f(x) = x^2 + 2$ is all real numbers. The domain for the composite function $g(f(x)) = \sqrt{1-x^2}$ is $-1 \leq x \leq 1$. The input function f(x) has no restrictions, so the domain of $g(f(x))$ is determined only by the composite function.

3. Find $f \circ g$ and $g \circ f$ and the domain of each, where $f(x) = x^2 + 2$ and $g(x) = \sqrt{7-x}$.

f ∘ g: Step 1. What is the domain of the inside function $g(x)$?   **x ≤ 7 Keep this!**

Step 2. The composite $f(g(x)) = \left(\sqrt{7-x}\right)^2 + 2 = 7 - x + 2 = 9 - x$

This function puts <u>no additional</u> restrictions on the domain, so the composite domain is **x ≤ 7**.

g ∘ f: Step 1. What is the domain of the inside function $f(x)$?   **All real numbers.**

Step 2. The composite $g(f(x)) = \sqrt{7 - (x^2 + 2)} = \sqrt{5 - x^2}$.

This function creates new restrictions: the composite domain is **-5 ≤ x ≤ 5.**

4. Find $f \circ g$ and $g \circ f$ and the domain of each, where $f(x) = \dfrac{1-x}{3x}$ and $g(x) = \dfrac{1}{1+3x}$.

f ∘ g: Step 1. What is the domain of the inside function $g(x)$?   **x ≠ -1/3 Keep this!**

Step 2. The composite $f(g(x)) = \dfrac{1 - \left(\dfrac{1}{1+3x}\right)}{3\left(\dfrac{1}{1+3x}\right)} = \dfrac{\dfrac{3x}{1+3x}}{\dfrac{3}{1+3x}} = x$

This function puts <u>no additional</u> restrictions on the domain, so the composite domain is **x ≠ -1/3.**

g ∘ f: Step 1. What is the domain of the inside function $f(x)$?   **x ≠ 0   Keep this!**

Step 2. The composite $g(f(x)) = \dfrac{1}{1+3\left(\dfrac{1-x}{3x}\right)} = \dfrac{1}{\dfrac{1}{x}} = x$

This function puts <u>no additional</u> restrictions on the domain, so the composite domain is **x ≠ 0.**

5. Find $f \circ g$ and $g \circ f$ and the domain of each, where $f(x) = \dfrac{3x}{x-1}$ and $g(x) = \dfrac{2}{x}$.

f ∘ g: Step 1. What is the domain of the inside function $g(x)$?   **x ≠ 0   Keep this!**

Step 2. The composite $f(g(x)) = \dfrac{3\left(\dfrac{2}{x}\right)}{\left(\dfrac{2}{x}\right)-1} = \dfrac{\dfrac{6}{x}}{\dfrac{2-x}{x}} = \dfrac{6}{2-x}$ Domain: **x ≠ 2**

Combine this domain with the domain from Step 1: the composite domain is **x ≠ 0 and x ≠ 2.**

g∘f: Step 1. What is the domain of the inside function f(x)?   **x ≠ 1   Keep this!**

Step 2. The composite $g(f(x)) = \dfrac{2}{\left(\dfrac{3x}{x-1}\right)} = \dfrac{2(x-1)}{3x}$ Domain: **x ≠ 0**

Combine this domain with the domain from Step 1: the composite domain is **x ≠ 1 and x ≠ 0.**

Example of Intersection of Domains

6. Find f∘g and g∘f and the domain of each, where $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt{x^2-1}$

f∘g: Step 1. What is the domain of the inside function g(x)? $x^2 \geq 1 \rightarrow$ **x ≥ 1 or x ≤ -1**

Step 2. The composite $f(g(x)) = \sqrt{\sqrt{x^2-1}-2}$. The domain of this function is where
$\sqrt{x^2-1} \geq 2 \rightarrow x^2-1 \geq 4 \rightarrow x^2 \geq 5 \rightarrow$ **x ≥ $\sqrt{5}$ or x ≤ -$\sqrt{5}$**

This function has a more restrictive domain than g(x), so the composite domain is
x ≥ $\sqrt{5}$ or x ≤ -$\sqrt{5}$. (This is the *intersection* of the two domains.)

g∘f: Step 1. What is the domain of the inside function f(x)?   **x ≥ 2**

Step 2. The composite $g(f(x)) = \sqrt{\left(\sqrt{x-2}\right)^2-1} = \sqrt{x-2-1} = \sqrt{x-3}$.   Domain: **x ≥ 3**

This function has a more restrictive domain than f(x), so the composite domain is **x ≥ 3.**
(This is the *intersection* of the two domains.)

Additional Problems

Find f∘g, g∘f, and the domain of each for the following functions.

1. $f(x) = x+3$ $g(x) = \sqrt{9-x^2}$ 4. $f(x) = x^2+2$ $g(x) = \sqrt{x-5}$

2. $f(x) = \sqrt{x+3}$ $g(x) = 2x-5$ 5. $f(x) = \dfrac{2}{x-3}$ $g(x) = \dfrac{5}{x+2}$

3. $f(x) = \dfrac{-3}{x}$ $g(x) = \dfrac{x}{x-2}$ 6. $f(x) = \dfrac{1}{\sqrt[3]{x-2}}$ $g(x) = x^2-3$

Answers

1. $f(g(x)) = \sqrt{9-x^2} + 3$ domain: $-3 \leq x \leq 3$

   $g(f(x)) = \sqrt{-x^2-6x}$ domain: $-6 \leq x \leq 0$

2. $f(g(x)) = \sqrt{2x-2}$ domain: $x \geq 1$

   $g(f(x)) = 2\sqrt{x+3} - 5$ domain: $x \geq -3$

3. $f(g(x)) = \dfrac{-3(x-2)}{x}$ domain: $x \neq 2$ and $x \neq 0$

   $g(f(x)) = \dfrac{3}{3+2x}$ domain: $x \neq 0$ and $x \neq -3/2$

4. $f(g(x)) = x - 3$ domain: $x \geq 5$

   $g(f(x)) = \sqrt{x^2-3}$ domain: $x \geq \sqrt{3}$ or $x \leq -\sqrt{3}$

5. $f(g(x)) = -\dfrac{2(x+2)}{3x+1}$ domain: $x \neq -2$ and $x \neq -1/3$

   $g(f(x)) = \dfrac{5(x-3)}{2x-4}$ domain: $x \neq 3$ and $x \neq 2$

6. $f(g(x)) = \dfrac{1}{\sqrt[3]{x^2-3}-2}$ domain: $x \neq \pm\sqrt{11}$

   $g(f(x)) = \dfrac{1}{\left(\sqrt[3]{x}-2\right)^2} - 3$ domain: $x \neq 8$